1 Optimum Receiver Principles

1.1 Transmitter

The communication model we are concerned with is shown in the following figure.

Source: The source is an *iid* process $\{V_n\}$ with alphabet

$$\Omega_V = \{v_1, v_2, \cdots, v_M\}$$

and probability mass function $p_V(v_i),\ i = 1, 2, , \cdots, M$. We assume that the source emits symbols at the rate of one symbol every T seconds. The source rate is $R = 1/T$ source symbols per second.

Modulator: The modulator is specified by the signal set

$$\mathcal{S} = \{s_1(t), s_2(t), \cdots, s_M(t)\}$$

where each $s_i(t) = 0$ for $t \notin [0, T]$. For the interval $[nT, (n+1)T]$, if the source put outs out the symbol $V_n = v_k$, the modulator produces the signal $s_k(t - nT)$ in in response to the source's output. This is called a *symbol by symbol modulation.*

Channel: The channel is the AWGN channel with noise process $\{N_w(t)\}$ having zero mean and the power spectral density $S_{N_W}(f) = \frac{N_0}{2}$.

Therefore for $t \in [nT, (n+1)T]$, the received signal is

$$r(t) = s(t) + N_W(t)$$

where $s(t) \in \mathcal{S}$.

1.2 Optimum Demodulator

The goal of the demodulator is to make a decision about $\{V_k\}$ based on $R(t)$, $t \in [kT, (k+1)T]$, where

$$R(t) = s(t) + N_W(t)$$

We wish our demodulator to have small average error probability, i.e.,

$$P(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} P(V_k \neq \hat{V}_k)$$

is to be small. It can be shown that since the source is independent and the noise is white, it is sufficient to consider a modulator that minimizes the probability of error for each source symbol, i.e.,

$$P(E) = P(V_k \neq \hat{V}_k) \quad \text{for all } k$$

1.3 Best decision rule

We observe $R(t)$ for $t \in (-\infty, \infty)$ and want to make a decision about each V_k.

Symbol be symbol detection

In a memoryless modulation system with an iid message sequence and AWGN channel, it is sufficient to use only the received signal $R(t)$ for $t \in [nT, (n+1)T]$ to make a decision about V_n.

Hereafter we use only the received signal $R(t)$ for $t \in [0, T]$ to make a decision about V_0, which we will denote as V.

According to the MAP rule we let

$$g[R(t)] = v_i$$

if and only if

$$p_{V|R(t)}(v_i | r(t), t \in [0, T])$$

is largest among all messages $v_k \neq v_i$.

We can't evaluate this probability because $R(t)$ is a waveform, i.e., random process. Therefore we look for a sufficient statistic for decisions about V based on $R(t)$ (a simplifying procedure).

Suppose we are given a set of functions

$$\{\phi_1(t), \phi_1(t), ..., \phi_L(t)\}$$

such that

1. (a) $\phi_i(t) = 0$ if $t \notin [0, T]$
   (b) $\int_0^T |\phi_i(t)|^2 dt < \infty$.

2. Each $s_k(t)$ can be expressed as a linear combination of the set

$$\{\phi_1(t), \phi_1(t), ..., \phi_L(t)\}$$

i.e.,

$$s_k(t) = \sum_{i=1}^{L} a_{ki}\phi_i(t)$$

Now given $R(t)$ compute

$$\mathbf{R} = (R_1, R_2, ..., R_L)$$

where

$$R_k = \int_0^T R(t)\phi_k(t)dt$$

Claim: $\mathbf{R}$ is a sufficient statistic for decisions about V based on

$$R(t), \quad 0 \leq t \leq T$$

This claim will be proved later.

Now we use $\mathbf{R}$ to estimate V. The best decision rule for V based on $\mathbf{R}$ is:

Given

$$\mathbf{R} = \mathbf{r}$$

let

$$g(\mathbf{r}) = m_i$$

if and only if

$$p_{X|\mathbf{R}}(m_i|\mathbf{r}) \geq p_{X|\mathbf{R}}(v_k|\mathbf{r}) \text{ for all } k \neq i$$

with ties broken arbitrarily.

Equivalently,

Given

$$\mathbf{R} = \mathbf{r}$$

let

$$g(\mathbf{r}) = v_i$$

if and only if

$$p_X(v_i)p_{\mathbf{R}|X}(\mathbf{r}|v_i) \geq p_X(v_k)p_{\mathbf{R}|X}(\mathbf{r}|v_k) \quad \text{for all} \quad k \neq i$$

For this we need to calculate $p_{\mathbf{R}|X}(\mathbf{r}|v_i)$ for all i.

Now given that $X = v_i$, $s_i(t)$ is transmitted and

$$R(t) = s_i(t) + N_W(t)$$

Thus

$$R_k = \int_0^T R(t)\phi_i(t)dt \tag{1}$$
$$= \int_0^T s_i(t)\phi_k(t)dt + \int_0^T N_W(t)\phi_k(t)dt \tag{2}$$

Then

$$\mathbf{R} = \mathbf{s}_i + \mathbf{N}$$

where

$$\mathbf{s}_i = (s_{i1}, s_{i1}, ..., s_{iL})$$

and

$$\mathbf{N} = (N_1, N_2, ..., N_L)$$

where

$$s_{ik} = \int_0^T s_i(t)\phi_k(t)dt$$

and

$$N_k = \int_0^T N_W(t)\phi_k(t)dt$$

Therefore,

$$p_{\mathbf{R}|X}(\mathbf{r}|v_i) = p_{\mathbf{N}}(\mathbf{r} - \mathbf{s}_i)$$

Thus we need to find $p_{\mathbf{N}}(\alpha)$.

Now $\mathbf{N}$ is a Gaussian random vector. We compute

1. $E[N_i] = 0$.
2. $$E[N_iN_j] = \frac{N_0}{2}\int_0^T \phi_i(t)\phi_j(t)dt$$

Everything becomes simpler if we choose $\phi_i(t)$'s so that the following conditions are satisfied.

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

We then get

$$\text{cov}(N_i, N_j) = E[N_i N_j] = \begin{cases} \frac{N_0}{2} & i = j \\ 0 & i \neq j \end{cases}$$

Thus

$$p_{\mathbf{N}}(\mathbf{x}) = \prod_{j=1}^{L} p_{N_j}(x_j) = \frac{1}{(\pi N_0/2)^{L/2}}\ e^{-\frac{\sum_{j=1}^{L} x_j^2}{N_0/2}}$$

and,

$$p_{\mathbf{R}|X}(\mathbf{r}|v_i) = \frac{1}{(\pi N_0/2)^{L/2}}\ e^{-\frac{\sum_{j=1}^{L}(r_j - s_{ij})^2}{N_0/2}}$$

The best decision rule is now given by:

Given

$$\mathbf{R} = \mathbf{r}$$

let

$$g(\mathbf{r}) = v_i$$

if and only if

$$\sum_{j=1}^{L}(r_j - s_{ij})^2 - \frac{N_0}{2}\ln p_X(v_i) \leq \sum_{j=1}^{L}(r_j - s_{kj})^2 - \frac{N_0}{2}\ln p_X(v_k) \text{ for all } k \neq i$$

Note that

$$\sum_{j=1}^{L}(r_j - s_{ij})^2 = ||\mathbf{r} - \mathbf{s}_i||^2$$

is the Euclidean distance between the two vectors $\mathbf{r}$ and $\mathbf{s}_i$.

The receiver structure is now as follows:

Figure for optimal receiver

The following questions remain to be resolved.

1. How to find the set of functions

$$\{\phi_1(t), \phi_2(t), ..., \phi_L(t)\}$$

so that they satisfy the following properties.

(a)

$$\int_0^T |\phi_i(t)|^2 \, dt = 1, \quad \forall i = 1, 2, \cdots L \tag{3}$$

$$\int_0^T \phi_i(t)\phi_j(t) \, dt = 0 \quad \forall i \neq j \tag{4}$$

(b) Each $s_k(t)$ can be expressed as a linear combination of the set

$$\{\phi_1(t), \phi_2(t), ..., \phi_L(t)\}$$

i.e.,

$$s_k(t) = \sum_{i=1}^{L} a_{ki}\phi_i(t)$$

Note:

- A function that satisfies the property in (3) is said to be normalized.
- A set of functions that satisfy the property (4) are said to be orthogonal.
- A set of functions satisfying (3) and (4) are said to be orthonormal (O.N.).

2. How to find the a_{ki}'s.

3. Show that **R** is a sufficient statistic.

4. Simplify receiver, equivalent implementation.

5. Compute $P(E)$.

6. Compare $P(E)$ for different signal sets, each with its own optimum receiver.

We first give the answer to question 2 since it is the easiest to answer. By our assumption on $\{\phi_i(t)\}$,

$$s_{kj} = a_{kj}$$

Proof:

$$\begin{aligned} s_{kj} &= \int_0^T s_k(t)\phi_j(t)\ dt = \int_0^T \left[\sum_{i=1}^L a_{ki}\phi_i(t)\right]\phi_j(t)\ dt \\ &= \sum_{i=1}^L a_{ki}\int_0^T \phi_i(t)\phi_j(t)\ dt = a_{kj} \end{aligned}$$